METHODES d'INTEGRATION

Note : $\frac{d}{dx} f(x) = f'(x)$

$\to d f(x) = f'(x)\,dx$

expl : ① $f(x) = x^3$

$\to dx^3 = 3x^2 dx$

② $d(x^3+2) = 3x^2 dx$

③ $d(x^3-7) = 3x^2 dx$

CCL : $\int 3x^2 dx = \begin{cases} x^3 & ① \\ x^3+2 & ② \\ x^3-7 & ③ \end{cases}$

$$\boxed{\int f'(x)\,dx = f(x) + k}$$

$k \in \mathbb{R}$

Formes élémentaires classiques

1) $\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

2) $\int a.f(x)\,dx = a.\int f(x)\,dx$

3) $\int dx = \int 1\,dx = x + k$

4) $\int x^m dx = \frac{x^{m+1}}{m+1} + k \quad (m \neq -1)$

5) $\int \frac{1}{x} dx = \ln(|x|) + k$

6) $\int a^x dx = \frac{a^x}{\ln a} + k$

7) $\int e^x dx = e^x + k$

8) $\int \sin x\,dx = -\cos x + k$

9) $\int \cos x\,dx = \sin x + k$

10) $\int \frac{1}{\cos^2 x} dx = \tan x + k$

11) $\int \frac{1}{\sin^2 x} dx = -\cotg x + k$

12) $\int \tan x\,dx = -\ln|\cos x| + k$

13) $\int \cotg x\,dx = \ln|\sin x| + k$

14) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}.\arctan\frac{x}{a} + k$

15) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$ quand $x^2 > a^2$

16) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$ quand $x^2 < a^2$

17) $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + k$

18) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + k$

Ajustement Linéaire

$\int f(ax+b)\,dx = \frac{1}{a}\int f(ax+b)\,dax$

$F' = f$

$= \frac{1}{a}\int f(ax+b)\,d(ax+b)$

$= \frac{1}{a}\int f(y)\,dy = \frac{1}{a}F(y) + k$

$= \frac{1}{a}F(ax+b) + k$

exemple : $\int \sin(\pi - 3x)\,dx$

$= -\frac{1}{3}\int \sin(\pi-3x)\,d(\pi-3x)$

$= -\frac{1}{3}\int \sin(y)\,dy$

$= \frac{1}{3}\cos(y) + k$

$= \frac{1}{3}\cos(\pi - 3x) + k.$

Changement de variable ou SUBSTITUTION.

$\int g'(x).f(g(x))\,dx = F(g(x)) + k$

avec $F' = f$.

exemples : ① $\int 3x^2.e^{x^3} dx$

$= \int e^{x^3} dx^3 = e^{x^3} + k$

② autrement écrit : $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

on pose : $u = \sqrt{x}$

(car $u' = \frac{1}{2\sqrt{x}}$)

$\to du = \frac{1}{2\sqrt{x}} dx$

$2\,du = \frac{1}{\sqrt{x}} dx$

donc $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int e^u.2\,du$

$= 2\int e^u du$

$= 2e^u + k$

$= 2e^{\sqrt{x}} + k$

③ $\int \frac{2x}{x^4+1} dx = \arctan(x^2) + k$

on a posé : $u = x^2$

Intégration par Partie

$\int f(x).g'(x)\,dx = f(x).g(x) - \int f'(x)g(x)\,dx$

en plus concis : $\int u\,dv = u.v - \int v\,du$

Exemples : ① $\int x.\cos x\,dx$

$= \int x\,d\sin x$

$= x.\sin x - \int \sin x\,dx$

$= x.\sin x + \cos x + k$

② $\int x.\ln x\,dx = \int \ln x\,d\frac{x^2}{2}$

$= \frac{x^2}{2}.\ln x - \int \frac{x^2}{2}\,d\ln x$

$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}.\frac{1}{x} dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + k.$

③ $\int x e^{ax} dx = x.\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} dx$

$= \frac{x.e^{ax}}{a} - \frac{e^{ax}}{a^2} + C$

$= \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right) + C$

④ $\int x^2 e^{ax} dx = x^2\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}.2x\,dx$

$= \frac{x^2 e^{ax}}{a} - \frac{2}{a}\underbrace{\int x.e^{ax} dx}_{③}$

$= \frac{x^2 e^{ax}}{a} - \frac{2e^{ax}}{a^2}\left(x - \frac{1}{a}\right) + C$

$= \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + C$

— INTEGRATION de FCT. TRIGO. —

utilisation possible d'identités trigonométrique.

$$\sin^2 x = \frac{1}{2}\left(1-\cos(2x)\right) \quad (1)$$

$$\cos^2 x = \frac{1}{2}\left(1+\cos(2x)\right) \quad (2)$$

$$\sin x . \cos x = \frac{1}{2}\sin(2x) \quad (3)$$

$$\frac{1}{\cos^2 x} = 1+\tan^2 x \quad (4)$$

$$\frac{1}{\sin^2 x} = 1+\cot g^2 x$$

Exemples : (1) $\int \cos^2 x\, dx$

$$= \int \frac{1}{2}\left(1+\cos(2x)\right)dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + k$$

(2) $\int \sin(x)\cos(x)\, dx$

$$= \frac{1}{2}\int \sin(2x)\, dx = \frac{1}{4}\cos 2x + k$$

(3) $\int \tan^2 x\, dx = \int\left(\frac{1}{\cos^2 x} - 1\right)dx$

$$= \tan x - x + k$$

(4) $\int \cos^3 x\, dx = \int \cos^2 x . \cos x\, dx$

$$= \int (1-\sin^2 x)\cos x\, dx$$

$$= \int \cos x\, dx - \int \sin^2 x \cos x\, dx$$

$$= \sin x - \int \sin^2 x\, d\sin x$$

$$= \sin x - \frac{\sin^3 x}{3} + k.$$

Autres "cas" trigonométriques.

I. $\int \sin^m x . \cos^n x\, dx$

Exemples : (1) $\int \sin^2 x . \cos^5 x\, dx$

$$= \int \sin^2 x \cos^4 x \cos x\, dx$$

$$= \int \sin^2 x\,(1-\sin^2 x)^2\, d\sin x$$

$$= \int u^2(1-u^2)^2 du$$

$$= \int (u^2 - 2u^4 + u^6) du$$

$$= \frac{u^3}{3} - \frac{2}{5}u^5 + \frac{u^7}{7} + k$$

$$= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + k$$

(2) $\int \sin^5 x\, dx = \int \sin^4 x . \sin x\, dx$

$$= -\int \sin^4 x\, d\cos x$$

$$= -\int (1-\cos^2 x)^2 d\cos x$$

$$= -\int (1-u^2)^2 du$$

$$= -\int 1 - 2u^2 + u^4\, du$$

$$= -u + \frac{2u^3}{3} - \frac{u^5}{5} + k$$

$$= -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + k$$

II. $\int \tan^n x\, dx$

Méthode : écrire $\tan^n x = \tan^{n-2} x . \tan^2 x$

Exemples : (1) $\int \tan^4 x\, dx$

$$= \int \tan^2 x . \left(\frac{1}{\cos^2 x} - 1\right) dx$$

$$= \int \tan^2 x . \frac{1}{\cos^2 x} dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x\, d\tan x - \int \tan^2 x\, dx$$

$$= \frac{\tan^3 x}{3} - \int\left(\frac{1}{\cos^2 x} - 1\right)dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x + k$$

(2) $\int \tan^3 v\, dv = \int \tan v . \tan^2 v\, dv$

$$= \int \tan v\left(\frac{1}{\cos^2 v} - 1\right)dv$$

$$= \int \tan v\, d\tan v - \int \tan v\, dv$$

$$= \frac{\tan^2 v}{2} + \ln|\cos v| + k.$$

— SUBSTITUTION TRIGONO —

Expressions contenant :

$\sqrt{a^2 - x^2}$ ou $\sqrt{x^2 \pm a^2}$

Quand $\sqrt{a^2-x^2}$ $\Rightarrow$ on pose $x = a\sin y$

Quand $\sqrt{a^2+x^2}$ $\Rightarrow$ on pose $x = a\tan y$

Quand $\sqrt{x^2-a^2}$ $\Rightarrow$ on pose $x = \frac{a}{\cos y}$

Exemples : (1) $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \ldots$

$x = a.\sin y$

$dx = a\cos y\, dy$

$$\ldots = \int \frac{a\cos y\, dy}{\sqrt{(a^2 - a^2\sin^2 y)^3}} = \int \frac{a\cos y\, dy}{a^3\cos^3 y}$$

$$\ldots = \frac{1}{a^2}\int \frac{1}{\cos^2 y} dy$$

$$= \frac{1}{a^2}\tan y + c$$

$$= \frac{1}{a^2}\tan\left(\arcsin\frac{x}{a}\right) \quad (*)$$

$$= \frac{x}{a^2\sqrt{a^2-x^2}} + k.$$

(*) voir chapitre cyclométrique

(2) $\int \frac{dx}{x\sqrt{4x^2+9}} = \int \frac{dx}{x\sqrt{(2x)^2+3^2}}$ (*)

d'où $a = 3$ et $u = 2x$

(*) devient : $\int \frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2+a^2}}$

$$= \int \frac{du}{u\sqrt{u^2+a^2}}$$

on pose $\begin{cases} u = a\tan y \\ du = \frac{a}{\cos^2 y}dy \end{cases}$

$$= \int \frac{\frac{a\, dy}{a.\tan y . \cos^2 y}}{\frac{a}{\cos y}}$$

$$= \frac{1}{a}\int \frac{1}{\sin y} dy$$

A voir

$$= \frac{1}{a}\ln\left|\frac{\sin y}{1+\cos y}\right| + k$$

A voir

$$= \frac{1}{a}\ln\left|\frac{\sqrt{u^2+a^2} - a}{u}\right| + k$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9} - 3}{2x}\right| + k$$

[3]

— INTEGRALE DEFINIE —

ex 1) Trouver $\int_1^4 x^2\,dx$

SOL: $\int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 = \frac{64}{3} - \frac{1}{3} = 21$

RETENIR! $\int_a^b f(x)dx = F(b)-F(a) = F(x)\Big|_a^b$

ex 2) Trouver $\int_0^a \frac{dx}{a^2+x^2}$

SOL: $\int_0^a \frac{1}{a^2+x^2}\,dx$

$= \frac{1}{a}\cdot \arctan\frac{x}{a}\Big|_0^a$

$= \frac{1}{a}\arctan 1 - \frac{1}{a}\arctan 0$

$= \frac{\pi}{4a}$

— CHANGEMENT de BORNES —

ex: évaluer $\int_0^{16} \frac{\sqrt[4]{x}}{1+\sqrt{x}}\,dx$

SOL: on pose $x = z^4$

d'où $\begin{cases} dx = 4z^3 dz \\ \sqrt{x} = z^2\ ;\ \sqrt[4]{x} = z \end{cases}$

Pour les bornes: $x=0 \rightarrow x=16$

$\Downarrow \qquad \Downarrow$

$z=0 \rightarrow z=2$

Donc: $\int_0^{16} \frac{\sqrt[4]{x}}{1+\sqrt{x}}\,dx = \int_0^2 \frac{z\cdot 4z^3 dz}{1+z^2}$

$4\int_0^2 \frac{z^4}{1+z^2}\,dz$

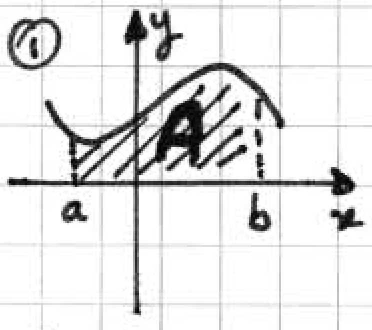
$= 4\int_0^2 \left[z^2 - 1 + \frac{1}{1+z^2}\right] dz$

$= 4\left(\frac{z^3}{3} - z + \arctan z\right)\Big|_0^2$

$= 4\left[\frac{8}{3} + 4\arctan 2\right]$

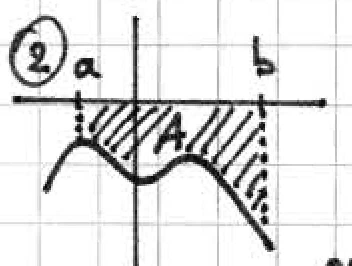
fin.

— CALCUL DES AIRES —

① 

$\forall x \in [a;b]: f(x) > 0$

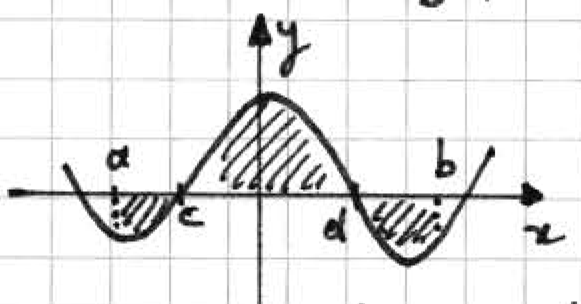
$A = \int_a^b f(x)dx$

② 

$\forall x \in [a;b]: f(x) < 0$

$A = -\int_a^b f(x)dx$

ou $A = \int_b^a f(x)\,dx$

③ 

$A = -\int_a^c f(x)dx + \int_c^d f(x)dx - \int_d^b f(x)dx$

DECOMPOSITION DE L'INTERVALLE D'INTEGRATION D'UNE ∫ DEFINIE

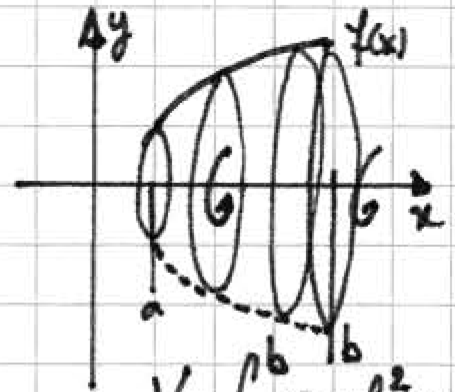
$\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^b f(x)dx$

$\forall x_1 \in [a;b]$

— ECHANGE de BORNES —

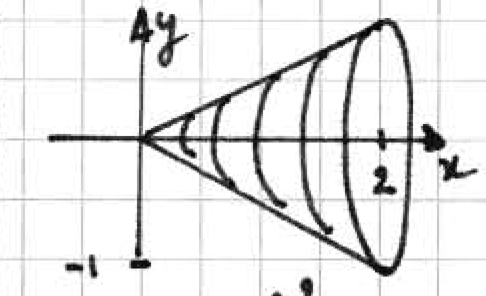
$\int_a^b f(x)dx = -\int_b^a f(x)dx$

— VOLUMES des SOLIDES de REVOLUTION —



$V = \int_a^b \pi f^2(x)\,dx$

Exemple: Volume d'un cône.



$f(x) = \frac{1}{2}x$

$V = \int_0^2 \pi \frac{x^2}{4}\,dx$

$= \pi \frac{x^3}{12}\Big|_0^2$

$= \frac{2\pi}{3}$

Remarques:

1) Volume d'un cône $= \frac{1}{3}B.h$ (Base)

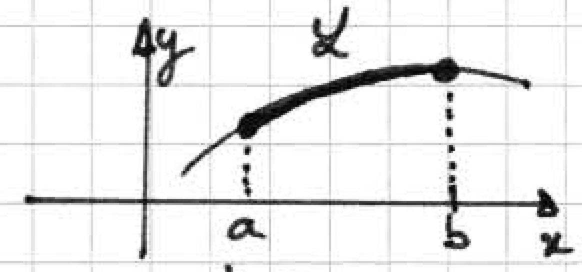
ici: $\begin{cases} h = 2 \\ B = \pi r^2\big|_{r=1} = \pi \end{cases}$

donc $V = \frac{1}{3}\pi.2 = \frac{2\pi}{3}$

2) Volume de révolution

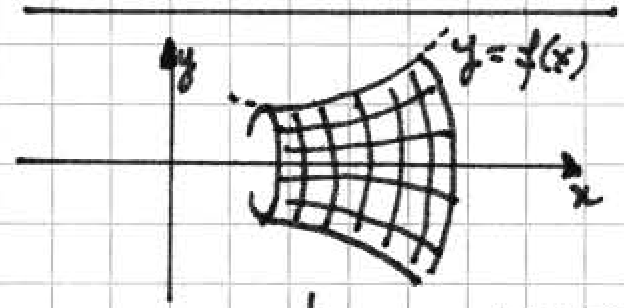
$V = \pi\int_a^b f^2(x)dx$

— LONGUEUR d'une COURBE —



$\mathcal{L} = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$

— AIRES des SURFACES de REVOLUTION —



$S = 2\pi\int_a^b f(x)\cdot\sqrt{1+\left(f'(x)\right)^2}\,dx$

Exemple: On fait tourner autour de Ox l'arc de la parabole cubique $y = x^3$ compris entre $x=0$ et $x=1$.

→ Calculer l'aire de la surface de révolution ainsi engendrée.

$S = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx$

$= \frac{\pi}{27}\left(\sqrt{(1+9x^4)^3}\right)\Big|_0^1$

$= \frac{\pi}{27}\left(10\sqrt{10} - 1\right) \simeq 3{,}6$

Remarque:

$\int x^3\sqrt{1+9x^4}\,dx$

$u = 1+9x^4$

$du = 36x^3dx$

$\frac{1}{36}du = x^3dx$

$= \int\sqrt{u}\,\frac{1}{36}du$

$= \frac{1}{36}\cdot\int u^{1/2}du = \frac{1}{36}\cdot\frac{2}{3}u^{3/2} + k = \frac{1}{54}(1+9x^4)^{\frac{3}{2}} + k$

EXERCICES

1. $\int \frac{dx}{x^2} =$
2. $\int \sqrt[3]{x^2}\,dx =$
3. $\int \frac{4x^2 - 2\sqrt{x}}{x}\,dx =$
4. $\int \sqrt{x}\,(3x-2)\,dx =$
5. $\int \frac{x^3 - 6x + 5}{x}\,dx =$
6. $\int \sqrt{a+bx}\,dx =$
7. $\int \frac{dy}{\sqrt{a-by}} =$
8. $\int (a+bt)^2\,dt =$
9. $\int x(2+x^2)^2\,dx =$
10. $\int y\,(a-by^2)\,dy =$
11. $\int \frac{4x^2}{\sqrt{x^3+8}}\,dx =$
12. $\int x^{m-1}\sqrt{1+x^m}\,dx =$
13. $\int \frac{(2+\ln x)}{x}\,dx =$
14. $\int \sin^2 x \,.\, \cos x\,dx =$
15. $\int \frac{\sin 2\varphi\,d\varphi}{\sqrt{\cos 2\varphi}}$
16. $\int \frac{e^x}{\sqrt{e^x-5}}\,dx$
17. $\int \frac{e^x + \sin x}{e^x - \cos x}\,dx =$
18. $\int \frac{1}{e^x}\,dx =$
19. $\int 10^x\,dx =$
20. $\int x . e^{x^2}\,dx =$
21. $\int \frac{e^x+4}{e^x}\,dx =$
22. $\int \frac{1}{\cos^2\theta\,\sqrt{1+2\tan\theta}}\,d\theta$

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23. $\int \frac{1}{x^2+9}\,dx =$
24. $\int \frac{1}{x^2-4}\,dx =$
25. $\int \frac{dy}{\sqrt{25-y^2}} =$
26. $\int \frac{dx}{\sqrt{16-9x^2}} =$
27. $\int \frac{e^x}{1+e^{2x}} =$
28. $\int \frac{dt}{(t-2)^2+9} =$
29. $\int \frac{x^3}{1-x^6}\,dx =$
30. Vérifier que : $\int \frac{1}{x^2+2x+5}\,dx = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + k$

31. $\int \frac{3}{x^2-8x+25}\,dx =$
32. $\int \frac{dx}{\sqrt{15+2x-x^2}} = \arcsin\left(\frac{x-1}{4}\right)+k$
A VERIFIER
33. $\int \frac{dx}{\sqrt{2x-x^2}} =$
34. $\int \frac{dx}{1+x+x^2} =$

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35. $\int \frac{1+2x}{1+x^2}\,dx =$
36. $\int \frac{x-1}{\sqrt{1-x^2}}\,dx =$

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37. $\int \cos^3 x \,.\, \sin x\,dx =$
38. $\int \frac{\sin^3\varphi}{\cos^2\varphi}\,d\varphi =$
39. $\int \sin^3 t \,.\, \cos^3 t\,dt =$
40. $\int \sin^2(ax)\,dx =$
41. $\int \sin^4 t\,dt =$
42. $\int \sin(2\theta) . \cos(4\theta)\,d\theta =$

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43. $\int x \sin x\,dx =$
44. $\int \ln(x)\,dx =$ $(dx = 1.dx)$
45. $\int \frac{u}{\cos^2 u}\,du =$

50. $\int v . \sin^2 v\,dv =$
51. $\int y^2 . \sin y\,dy =$
52. $\int x . a^x\,dx =$
53. $\int \arcsin x\,dx =$
54. $\int \arctan x\,dx =$
55. $\int \arccos(2x)\,dx =$
56. $\int x . \arctan x\,dx =$
57. $\int \frac{x^2}{e^x}\,dx =$
58. $\int e^\theta . \cos\theta\,d\theta =$
59. $\int \frac{\ln x}{(x+1)^2}\,dx =$
60. $\int \frac{x . e^x}{(1+x)^2}\,dx$

<u>Remarque</u>: l'intégration est une opération bien plus difficile que la dérivation. Il est, par exemple, impossible d'effectuer l'intégration $\int \sqrt{x} . \sin x\,dx$